14th Week Friday
26 Mayıs 2023 Cuma 10:28

Some Operations on Vectors $\to \in \mathbb{R}^n$

→ Dot Product (Inner Product) :

inputs → output

2 vectors from same $\mathbb{R}^n$ → a real number

$\vec{u} = (u_1, u_2, \ldots, u_n) \in \mathbb{R}^n$

$\vec{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$

$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n = \sum_{i=1}^{n} u_i v_i \in \mathbb{R}$

Ex/ $u = (1, -2, 3) \in \mathbb{R}^3$, $v = (3, 2, -5) \in \mathbb{R}^3$ $\Rightarrow$ $u \cdot v = \underbrace{1.3}_{3} + \underbrace{(-2)(2)}_{-4} + \underbrace{3(-5)}_{-15} = -16 \in \mathbb{R}$

Ex/ $\|\vec{u}\| = \sqrt{1+4+9} = \sqrt{14}$

$\|\vec{v}\| = \sqrt{9+4+25} = \sqrt{38}$

Norm of a Vector : (~ length)

$\|\vec{u}\| \to$ norm of the vector $\vec{u}$.

$\vec{u} = (u_1, u_2, \ldots, u_n) \in \mathbb{R}^n \Rightarrow \|\vec{u}\| = \sqrt{u_1^2 + u_2^2 + \ldots + u_n^2}$

! Norm ≥ 0 Norm $= 0 \Leftrightarrow \vec{u} = (0, 0, \ldots, 0)$

! $\vec{u} \cdot \vec{u} = u_1 . u_1 + u_2 . u_2 + \ldots + u_n v_n = \|\vec{u}\|^2$

Normed Vector (unit vector) ;

$\vec{u}_0 \to$ normed vector of u

$\vec{u} = (u_1, u_2, \ldots, u_n) \in \mathbb{R}^2$ $\vec{u}_0 = \frac{1}{\|\vec{u}\|} . \vec{u} = \left(\frac{u_1}{\|\vec{u}\|}, \frac{u_2}{\|\vec{u}\|}, \ldots, \frac{u_n}{\|\vec{u}\|} \right) \in \mathbb{R}^n$ $\|\vec{u}_0\| = 1$

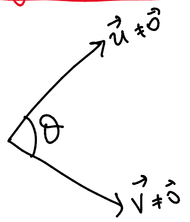
represents the direction of $\vec{u}$.

$\|\vec{u}_0\| = \sqrt{\frac{u_1^2 + u_2^2 + \ldots + u_n^2}{\|\vec{u}\|^2}} = \frac{\|\vec{u}\|}{\|\vec{u}\|} = 1$

Ex/ $\vec{u} = (1, -2, 3)$ $\|\vec{u}\| = \sqrt{1+4+9} = \sqrt{14}$

$\vec{u}_0 = \underbrace{\frac{1}{\|\vec{u}\|}}_{\text{scalar}} \vec{u} = \frac{1}{\sqrt{14}} . (1, -2, 3) = \left(\frac{1}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right)$

Angle Between Two Vectors

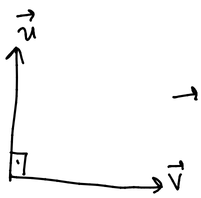


$\vec{u}, \vec{v} \in \mathbb{R}^n$

$$\boxed{\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| . \|\vec{v}\|}}$$

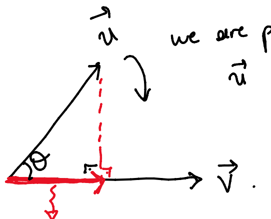
$\cos\theta = 0 \Leftrightarrow \vec{u} \cdot \vec{v}$

$0 < \theta < \pi$ $\cos\theta = 0 \Rightarrow \theta = \pi/2 \to \vec{u}$ and $\vec{v}$ are orthogonal.



→ $\vec{u}$ and $\vec{v}$ are orthogonal $\Leftrightarrow$ $\boxed{\vec{u} \cdot \vec{v} = 0}$

Projection :



we are projecting $\vec{u}$ onto $\vec{v}$

$\text{proj}_{\vec{v}}\vec{u}$ $\rightarrow$ length of this vector . direction

scalar component of the projection

$\|\vec{u}\|.\cos\theta$

$$\vec{v}_0 = \frac{\vec{v}}{\|\vec{v}\|}$$

$$\text{proj}_{\vec{v}}\vec{u} = \|\vec{u}\|\cos\theta \cdot \frac{\vec{v}}{\|\vec{v}\|} = \cancel{\|\vec{u}\|} \cdot \frac{\vec{u}\cdot\vec{v}}{\cancel{\|\vec{u}\|}\|\vec{v}\|} \cdot \frac{\vec{v}}{\|\vec{v}\|} = \underbrace{\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}}_{\in\mathbb{R}} \cdot \vec{v}$$

Ex/ $\vec{u} = (3,4)$ a) Find the angle between $\vec{u}$ and $\vec{v}$

$\vec{v} = (-1,7)$ b) Find $\text{proj}_{\vec{v}}\vec{u}$

a) $$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|.\|\vec{v}\|} = \frac{\cancel{25}}{\cancel{5}.\cancel{5}.\sqrt{2}} = \frac{1}{\sqrt{2}} \rightarrow \theta = \frac{\pi}{4}$$

$\vec{u}\cdot\vec{v} = (3,4)\cdot(-1,7) = 3(-1)+4.7 = 25$

$\|\vec{u}\| = \sqrt{3^2+4^2} = 5$

$\|\vec{v}\| = \sqrt{1^2+7^2} = \sqrt{50} = 5\sqrt{2}$

b) $$\text{proj}_{\vec{v}}\vec{u} = \|\vec{u}\|\cos\theta \cdot \frac{\vec{v}}{\|\vec{v}\|}$$

$$= \frac{5}{\sqrt{2}} \cdot \left(\frac{-1}{5\sqrt{2}}, \frac{7}{5\sqrt{2}}\right)$$

$$\text{proj}_{\vec{v}}\vec{u} = \left(\frac{-1}{2}, \frac{7}{2}\right)$$

Ex/ $\vec{u} = (5,2)$ a) Find $\text{proj}_{\vec{v}}\vec{u}$

$\vec{v} = (1,-3)$ b) Find the scalar component of $\text{proj}_{\vec{v}}\vec{u}$ $\rightarrow$ $\|\vec{u}\|\cos\theta$

it may have neg. sign.

a) $$\text{proj}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2} \cdot \vec{v} = \frac{-1}{10} \cdot \vec{v} = \left(\frac{-1}{10}, \frac{3}{10}\right)$$

$\vec{u}\cdot\vec{v} = 5.1+2.(-3) = -1$

$\|\vec{v}\|^2 = 1^2+(-3)^2 = 10$

b) $$\|\text{proj}_{\vec{v}}\vec{u}\| = \sqrt{\frac{1}{100}+\frac{9}{100}} = \frac{1}{\sqrt{10}}$$

$$\|\vec{u}\|\cos\theta = \cancel{\|\vec{u}\|} \frac{\vec{u}\cdot\vec{v}}{\cancel{\|\vec{u}\|}\|\vec{v}\|} = \frac{-1}{\sqrt{10}}$$

$$\text{proj}_{\vec{v}}\vec{u} \neq \text{proj}_{\vec{u}}\vec{v}$$

$\vec{u}\cdot\vec{v} = 0$. $\vec{u} \perp \vec{v}$

Orthogonal Spaces ($\mathbb{R}^n$)

$S \leqslant \mathbb{R}^n$, $T \leqslant \mathbb{R}^n$ → two subspaces of $\mathbb{R}^n$

$\forall \vec{s} \in S$ and $\forall \vec{t} \in T \Rightarrow \vec{s} \cdot \vec{t} = 0 \iff S$ and T are orthogonal spaces.

Orthogonal Complement of a Subspace: $S \leqslant \mathbb{R}^n$ $\quad S^{\perp} \to S$ dual, orthogonal complement of S.

$$S^{\perp} = \{ \vec{v} \in \mathbb{R}^n : \vec{v} \cdot \vec{s} = 0, \forall \vec{s} \in S \} \leqslant \mathbb{R}^n$$

Ex/ $\mathbb{R}^3 \to S = \text{span}(e_1) = \{ (r,0,0) : r \in \mathbb{R} \} \leqslant \mathbb{R}^3$

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$S^{\perp} = \{ (x,y,z) : \underbrace{(x,y,z) \cdot (r,0,0) = 0}_{xr = 0 \Rightarrow x = 0} \} = \{ (0,y,z) : y,z \in \mathbb{R} \}$$

$xr = 0 \Rightarrow x = 0 \quad y \to$ free $\quad z \to$ free

ex/ $(1,0,0) \in S \quad (0,-3,5) \in S^{\perp} \Rightarrow (1,0,0) \cdot (0,-3,5) = 0+0+0 = 0$